

Some Applications of Flexible Programming

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Abstract

A large number of problems in production, planning and scheduling, location, transportation, finance and engineering design may be cast into a linear programming framework. Inconsistency of constraints and unboundedness of the feasible set are of common occurrence while solving these problems. This is due to technological environmental and competitive factors which interact in a complicated fashion. While anti-cycling rules are implemented in most software the most used technique for solving linear programs (the Simplex method), nothing has been done against the vacuousness or the unboundedness of the feasible set. In this paper, we discuss a pragmatic approach for progressing toward a satisfying solution in line with the decision maker's temperament and system of values. At the heart of this approach lies in the use of the fuzzy paradigm to represent Decision maker's views and to handle them. Numerical examples are carried out for the sake of illustration.

Keywords

Linear Programming; Inconsistency; Unboundedness; Fuzzy Sets

Introduction

Linear programming is an important tool in the arsenal of means at a Decision Marker's disposal. Indeed it is one of the most frequently applied Operational research modals in solving real word problems [(R. J Vanderbei, 2008), (S. Nash, 1996), (G. Dantzig, 1951)]. Linear programming's theoretical underpinning is now well established and as a result, a broader array of techniques including the Simplex method (S. I Gass, 2003), the Ellipsoid method (D. Dubois, 2011) and the Interior point methods (N. Karmarkar, 1984) have been developed. User-friendly software with powerful computational and visualisation capabilities has also been pushed forward. While solving a real-life problem using linear programming, it may happen that either the constraints of the problem are inconsistent or the feasible set is not bounded. These irregularities pose serious challenges, to the Decision Maker, mostly when he is in need of a solution "here and now". While anti-cycling rules serve as guidelines in implementing efficient techniques for dealing with

degeneracy, there is notable lack of operational tools for guiding decision making in the presence of inconsistencies and unboundedness.

In this paper we take the option of facing these irregularities by abandoning the pure rationality principle (A. Schuts, 1943) and its naturalist trend of seeking an "optimum optimorum", that anyway does not exist in these cases.

Rather, we adopt a constructivist perspective of creating a satisfying solution (H. A Simon, 1997) in line with preferences and system value of the Decision Maker. At the heart of our approach lies in the fuzzy paradigm (L. A Zadeh, 1978) that helps in representing and handling Decision Maker's preferences.

For other uses of Fuzzy set Theory in Decision Science we refer the reader to [(L. G Khachiyan, 1979), (S. A Orlovsky, 1978)].

The remainder of this paper unfolds as follows. In the next section we briefly discuss how to check infeasibility and unboundedness for a linear program. In section 3 we present some ways for guiding decision making in the presence of inconsistency and unboundedness in a linear programming setting. Numerical examples aiming to illustrate the effectiveness of our approach are carried out in Section 4. We end up in Section 5 with some concluding remarks along with lines for further development, in the field.

Checking Infeasibility and Unboundedness in a Linear Programming Context

Before proceeding further, we find it convenient to discuss some ways for checking inconsistency of the constraints and unboundedness of the feasible set, of a linear program. An interested reader may consult [8] for more details on these matters.

Inconsistency Criterion

The following result due to Minkowski and Farkas (S.

Kutateladze, 2010) and bearing the latter's name will be of paramount importance in stating Inconsistency criterion that will be used in this paper.

1) Farkas Lemma

Let A be a given $n \times m$ real matrix and b given m vector. The inequality $b^T y \geq 0$ holds for all $y \in \mathbb{R}^n$ satisfying $Ay \geq 0$ if and only if there is an n vector $\delta \geq 0$ such that $A^T \delta = b$.

The proof of this result may be found elsewhere (M. Comforti, 2008).

A result that extends quite canonically the Farkas Lemma is known under the name of S-Lemma (I. Polik, 2007). The basic idea of this widely used mathematical result came from Control Theory but it has important consequences in quadratic and Semi-definite Optimization.

It is worthy nothing that Farka's Lemma is equivalent to the following statement:

The system

$$\begin{cases} A^T x = b \\ x \geq 0 \end{cases} \quad (1.1)$$

is incompatible if and only if the system

$$\begin{cases} b^T y < 0 \\ Ay \geq 0 \\ y \in \mathbb{R}^m \end{cases} \quad (1.2)$$

has a solution.

2) Feasibility Test for a Linear Program

Consider the following linear program

$$(P) \begin{cases} \min cx \\ Ax = b \\ x \geq 0 \end{cases}$$

It is not a loss of generality to ponder only equality constraints. As a matter of fact, by making use of slack variables, a system of constraint inequalities can be converted to a system of equalities.

From Section 2.1.1 we can derive the following inconsistency criterion for (P) .

Solve the system (1.2).

If this system has a solution then (P) is

inconsistent. Otherwise (P) is feasible.

Unboundedness Criterion

The following result due to Weyl tells us that the unboundedness case may occur.

1) Weyl's Theorem

Consider $X = \{x \in \mathbb{R}^n / Ax \leq b, x \geq 0\}$ where A and b

are as in Section 2.1.1. Then either X is empty or X is a bounded polyhedra or X is an unbounded polyhedra set.

For the proof of this result the reader may consult [12]

2) Case of Non-finite Optimal Solution

Consider (P) and let B be an $m \times m$ submatrix of A such that $\det B \neq 0$ i.e. B is a basis. Let $I(B)$ stands for the set of indexes of columns of B and

$$Z_j = \sum_{i \in I(B)} c_i a_{ij}$$

Denote by k the index of the basic variable that enters the basis i.e. k is such that

$$Z_k - c_k = \max_{Z_j - c_j > 0} (Z_j - c_j) \quad (1.3)$$

Then the following statements, the proof of which may be found in (S. Nash, 1996) hold true.

Proposition 1

If $Z_j - c_j \leq 0 \quad \forall j$, then the basic solution corresponding to the basis B is optimal.

Proposition 2

If for k defined in (1.3), $a_{ik} \leq 0 \quad \forall i$ then (P) has no finite optimal solution.

Dealing With Inconsistency and Unboundedness in a Linear Programming Framework

In an attempt to apply Linear Programming to concrete real-life problems, one may (according to the Weyl's Theorem) encounter situations in which the feasible set of the linear program under consideration is either empty or not boundless. In this case the

Decision maker will not be happy with inconsistency and unboundedness reports. Our claim here is that simple and intuitive ideas from Fuzzy set Theory may be used, with proper reason; to make a relaxation of constraints in the case the feasible set is empty. The same ideas may also help convert the original problem into a flexible one, in the case of unboundedness. Both approaches rely heavily on Decision Maker's views and opinions. In this way one enhances the degree of conformity between the evolutions of the decision-making process and the value system of those involved in the process. The ultimate objective is to deliver a satisfying solution instead of an optimal one, which anyway does not exist.

Resolving Inconsistencies among Constraints

1) Analysis

Consider the linear program

$$(P2) \begin{cases} \min cx \\ Ax \leq b \\ x \geq 0 \end{cases}$$

where A is an $n \times m$ real matrix, b an n vector, c an m -vector and $x \in R^m$. Suppose that the feasibility test (see Section 2.1.2) reveals that the feasible set of $(P1)$ is empty. Instead of sending an inconsistency report, one may attempt relaxations of the objective and the constraints in a way to find a satisfying solution. Such a relaxation is given as the following Flexible program:

$$(P1)' \begin{cases} \widetilde{\min} cx \\ Ax \lesseqgtr b \\ x \geq 0 \end{cases}$$

Where $\sim$ means that $\min$ and $\leq$ are no longer strict imperatives but some leeways may be accepted. An interpretation which goes with $(P1)'$ is as follows.

Find $x \in X = \{x \in R^m, x \geq 0\}$ such that cx is as

well as possible below a reasonable level b_0 and such that the constraints

$$A_i x \leq b_i; i = 1, \dots, n \quad (1.4)$$

are met as well as possible. Or merely, find $x \in X$ such that the following system holds

$$A_i x \lesseqgtr b_i; i = 0, 1, \dots, n \quad (1.4)'$$

Where $A_0 = c$.

A convenient way to represent these soft constraints is through appropriate fuzzy sets of R , the membership functions of which are

$$\mu_i; i = 0, 1, \dots, n.$$

These membership functions may be defined as follows:

$$\mu_i(x) = \begin{cases} 1 & \text{if } A_i x \leq b_i \\ & \text{(i.e if the } i^{\text{th}} \text{ constraint of (1.4) is satisfied)}, \\ 1 - \frac{A_i x - b_i}{d_i} & \text{if } b_i < A_i x \leq b_i + d_i \\ & \text{(i.e if the } i^{\text{th}} \text{ constraint of (1.4) is violated to some acceptable extent)}, \\ 0 & \text{if } A_i x > b_i + d_i \\ & \text{(i.e if the } i^{\text{th}} \text{ constraint of (1.4) is strongly violated).} \end{cases}$$

Where d_i $i = 0, 1, \dots, n$ are fixed by the Decision Maker. They are subjectively chosen constants of admissible violations.

According to the Bellman-Zadeh's confluence principle (R. E Bellman L), a decision in a fuzzy environment is an option that has the highest membership degree in the fuzzy set intersection of the objective and the constraints of the problem.

This means we seek for $x^* = \arg \max \mu_D(x)$,

Where $\mu_D(x) = \bigwedge_{i=0}^n \mu_i(x)$, $\bigwedge$ standing for the connective

"and". Using the $\min T$ -norm (L. A Zadeh, 1978) to translate the meaning of the intersection, we have to solve the following optimization problem

$$(P2) \begin{cases} \max \min_{i=0, \dots, n} \mu_i(x) \\ x \in \bigcap_{i=0}^n \text{Supp} \mu_i \end{cases}$$

Where $Supp\mu_i$ stands for support of μ_i .

(P2) is equivalent to the following mathematical program:

$$(P2)' \begin{cases} \max \lambda \\ \lambda \leq 1 - \frac{A_i x - b_i}{d_i}; i = 0, 1, \dots, n \\ A_i x \leq b_i + d_i; i = 0, 1, \dots, n \\ x \geq 0 \end{cases}$$

This linear program can be solved by existing mathematical software like LINDO, LINGO, COMPLEX, XPRESS.

2) Description of the Method

From the foregoing discussion we obtain the following flowchart for dealing with inconsistency in a linear programming setting.

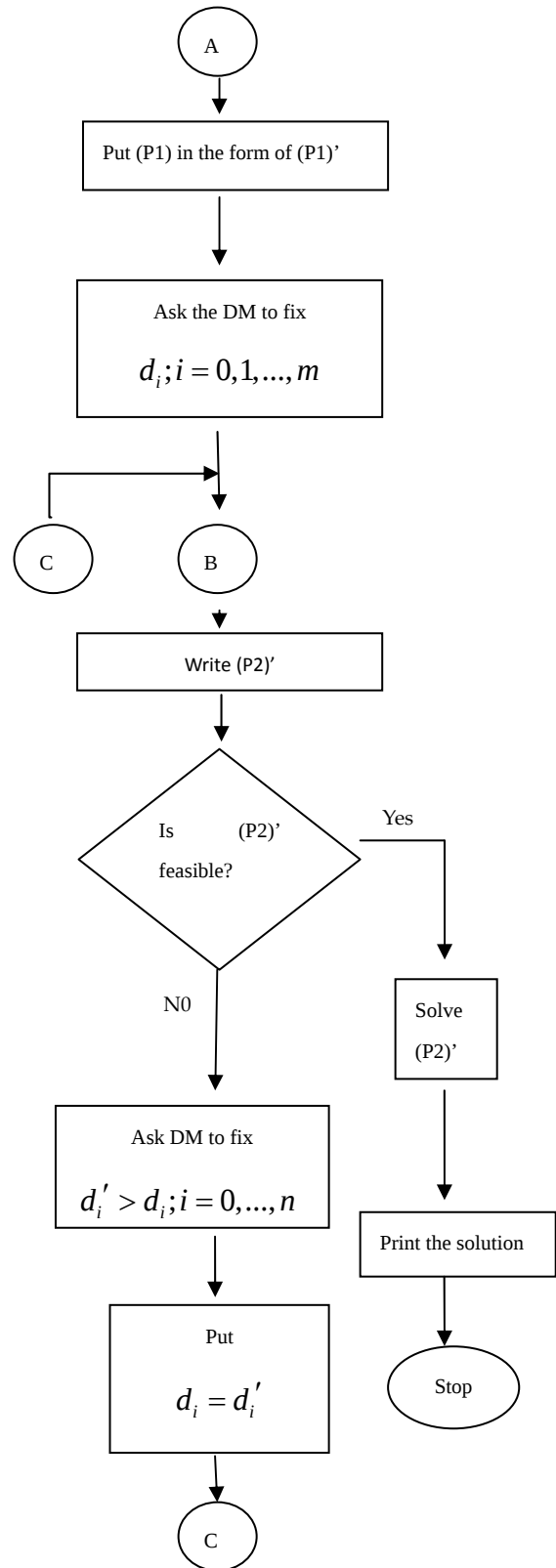
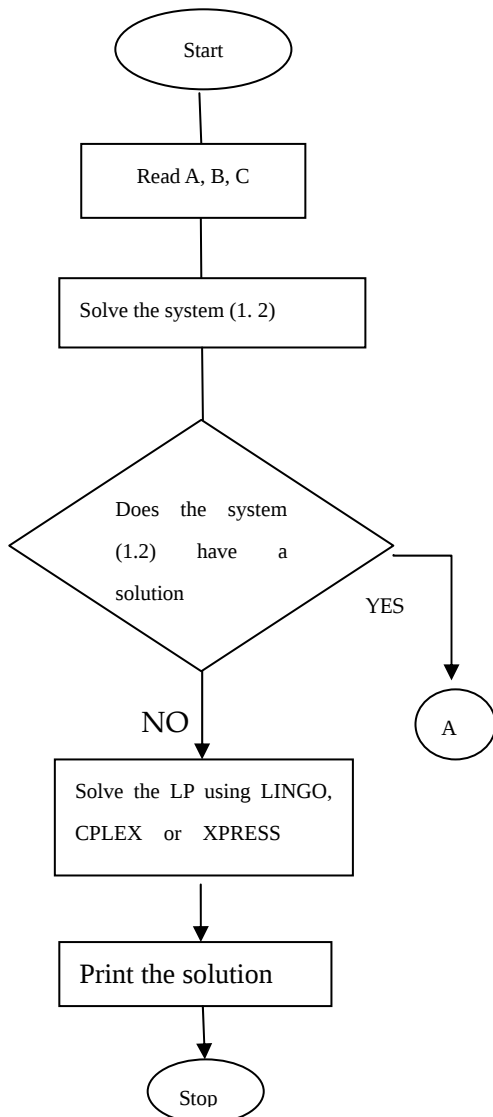


FIGURE 1 FLOWCHART OF LP METHOD IMPLEMENTING INFEASIBILITY

Handling Unboundedness of the Feasible Set

Consider the following linear program

$$(P3) \begin{cases} \max cx \\ Ax \leq b \\ x \geq 0 \end{cases}$$

where A is an $n \times m$ real matrix, b an n -vector, c an m -vector and $x \in \mathbb{R}^m$. Suppose that the criterion for detecting unboundedness (see Section 2.2.2) reveals that $(P3)$ has no finite optimal solution. Instead of delivering an unboundedness report, one may, with the help of the Decision Maker, reformulate $(P3)$ as follows:

$$\begin{cases} \text{Find} & x \in X = \{x \in \mathbb{R}^m / x \geq 0\} \\ \text{such that} & cx \text{ is sufficiently large} \\ \text{while } Ax \leq b & \text{is satisfied as well as possible} \end{cases}$$

Or merely as the following Flexible program:

$$(P3)' \begin{cases} \widetilde{\max} cx \\ Ax \leq b \\ x \geq 0 \end{cases}$$

Following Zimmerman (H. J Zimmermann, 1996), $(P3)'$ may be written as

$$(P3)'' \begin{cases} \text{Find } x \in X \text{ such that} \\ cx \gtrsim b_0 \\ A_i x \lesssim b_i; i=1, \dots, n \\ x \geq 0 \end{cases}$$

Where b_0 is a threshold provided by the Decision maker to calibrate his view on the fact that the objective function is large enough. These flexible constraints may be represented as fuzzy sets of X with membership functions:

$$\mu_0(x), \mu_1(x), \dots, \mu_n(x).$$

These membership functions should be determined by capturing the Decision maker's preferences.

For the sake of illustration assume that the decision maker agrees with the following piecewise linear

functions.

$$\mu_0(x) = \begin{cases} 0 & \text{if } cx \leq b_0 - d_0 \\ & \text{(i.e if } cx \text{ is too small),} \\ \frac{cx - (b_0 - d_0)}{d_0} & \text{if } b_0 - d_0 < cx \leq b_0 \\ & \text{(i.e if } cx \text{ is enough close to } b_0), \\ 1 & \text{if } cx > b_0 \\ & \text{(i.e if } cx \text{ is sufficiently large).} \end{cases}$$

and for $i=1, \dots, n$:

$$\mu_i(x) = \begin{cases} 1 & \text{if } A_i x \leq b_i \\ 1 - \frac{A_i x - b_i}{d_i} & \text{if } b_i < A_i x \leq b_i + d_i \\ 0 & \text{if } A_i x > b_i + d_i \end{cases}$$

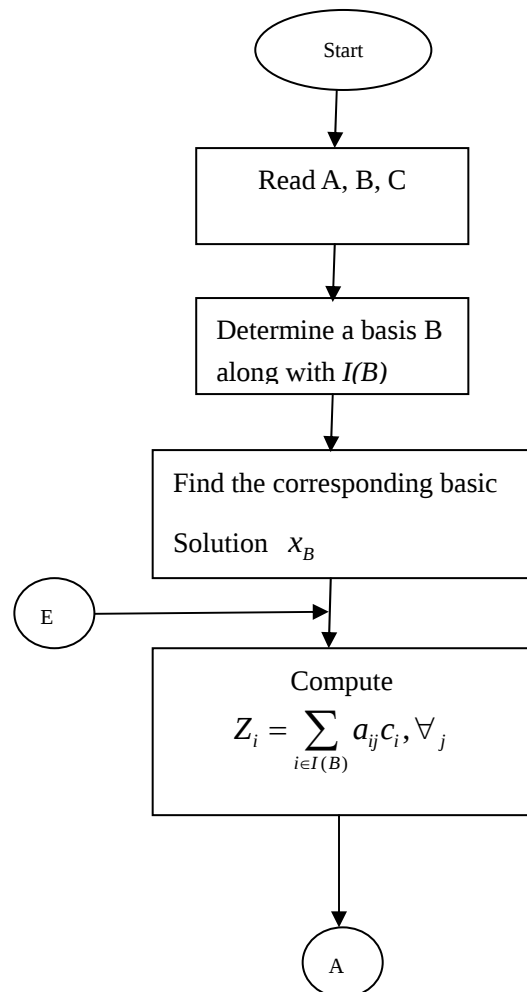


FIGURE 2 PROCEDURE FOR IMPLEMENTING UNBOUNDEDNESS IN A LP PROBLEM

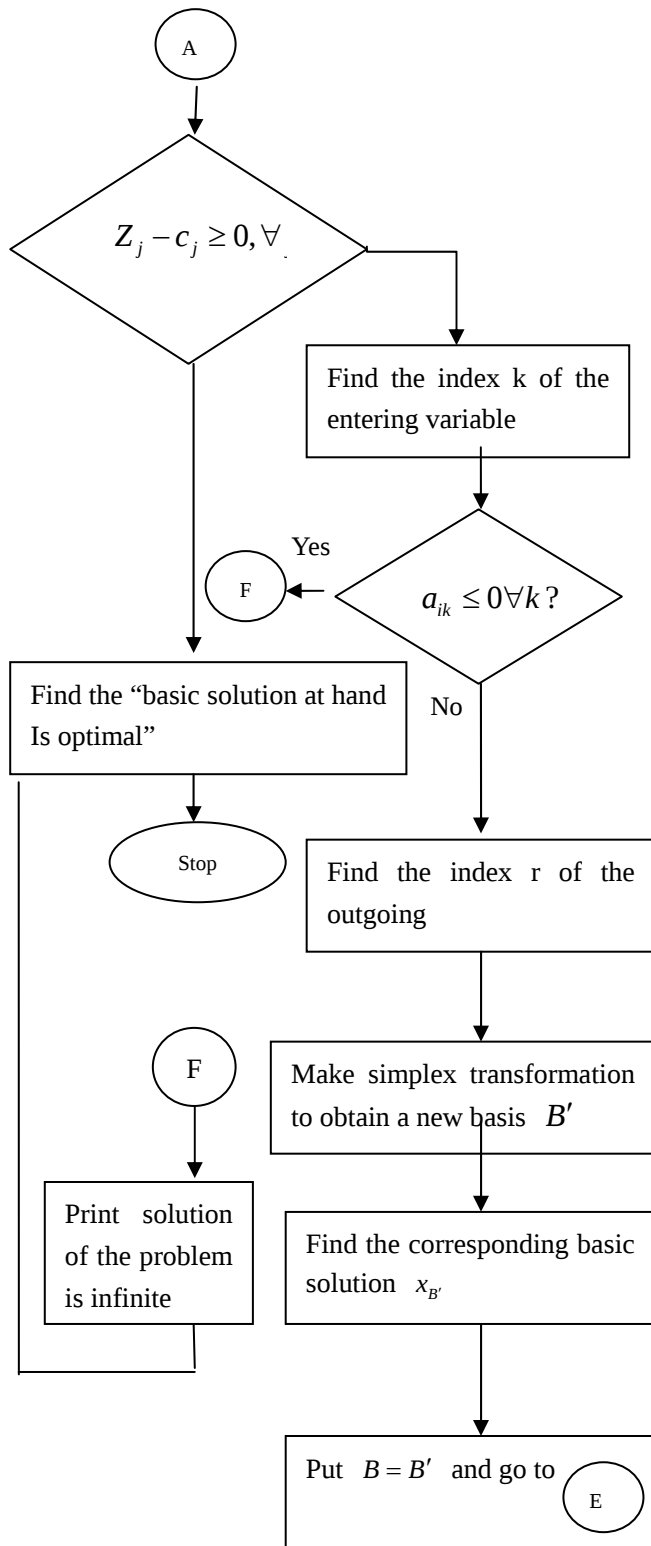


FIGURE 3 PROCEDURE FOR IMPLEMENTING UNBOUNDEDNESS IN A LP PROBLEM CONTINUED

$$x^* = \arg \max_{x \in X} \mu_D(x) \quad \text{where}$$

$$\mu_D(x) = \min_{i=0,1,\dots,n} \mu_i(x)$$

Applying Bellman-Zadeh's confluence principle like in

Section 3.1.1 and translating the intersection by the min operator, the solution to (P3) will be

Such an x^* may be obtained by solving the following mathematical program:

$$(P3)''' \begin{cases} \max \min_{i=0,1,\dots,n} \mu_i(x) \\ x \in \bigcap_{i=0}^n \text{Supp} \mu_i \end{cases}$$

(P3)''' is equivalent to the following optimization problem

$$(P3)^{iv} \begin{cases} \max \lambda \\ \lambda \leq \frac{cx - (b_0 - d_0)}{d_0} \\ \lambda \leq 1 - \frac{A_i x - b_i}{d_i}; i = 1, \dots, n \\ cx \geq b_0 - d_0 \\ A_i x \leq b_i + d_i; i = 1, \dots, n \\ x \geq 0 \end{cases}$$

The above considerations lead to the following procedure for implementing unboundedness in a linear programming framework.

Numerical Examples

Case of Inconsistent Constraints

Consider the linear program

$$(P_I) \begin{cases} \max (5x_1 + 4x_2) \\ x_1 + x_2 \leq 2 \\ -2x_1 - 2x_2 \leq -9 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

This program is infeasible. Indeed, the second constraints implies that

$$x_1 + x_2 \geq 4.5$$

Which contradicts the first constraints. Let now apply the method described in section 3.1 to reconcile theses inconsistent constraints. Assume that in the context under consideration the Decision Maker interprets

$\widetilde{\max} (5x_1 + 4x_2)$ as $5x_1 + 4x_2$ should be, as well as possible, greater than 10.

Then $(P_I)'$ reads:

Find $x \in X = \{(x_1, x_2) \in \mathbb{R}^2 / x_1 \geq 0, x_2 \geq 0\}$ such that:

$$5x_1 + 4x_2 \gtrsim 10 \quad (0)$$

$$x_1 + x_2 \lesssim 2 \quad (1)$$

$$-2x_1 - 2x_2 \lesssim -9 \quad (2)$$

Suppose further that the Decision Maker fix the respective constants for admissible violations of constraints (0)–(2)

As follows $d_0 = 5$, $d_1 = 1$, $d_2 = 4.5$.

Then the flexible constraints (0)–(2) may be represented as fuzzy sets of X the membership functions of which are respectively:

$$\mu_0(x) = \begin{cases} 1 & \text{if } 5x_1 + 4x_2 \geq 10 \\ \frac{5x_1 + 4x_2 - 5}{10 - 5} & \text{if } 5 \leq 5x_1 + 4x_2 < 10 \\ 0 & \text{if } 5x_1 + 4x_2 < 5 \end{cases}$$

$$\mu_1(x) = \begin{cases} 1 & \text{if } x_1 + x_2 \leq 2 \\ 1 - \frac{(x_1 + x_2) - 2}{(2 + 1) - 2} & \text{if } 2 < x_1 + x_2 \leq 3 \\ 0 & \text{if } x_1 + x_2 > 3 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & \text{if } -2x_1 - 2x_2 \leq -9 \\ 1 - \frac{-(2x_1 + 2x_2) + 9}{-9 + 4.5 + 9} & \text{if } -9 < -2x_1 - 2x_2 \leq -4.5 \\ 0 & \text{if } -2x_1 - 2x_2 > -4.5 \end{cases}$$

It is clear that the supports of the three fuzzy sets are respectively:

$$\text{Supp}(\mu_0) = \{x \in X / 5x_1 + 4x_2 \geq 5\}$$

$$\text{Supp}(\mu_1) = \{x \in X / x_1 + x_2 \leq 3\}$$

$$\text{Supp}(\mu_2) = \{x \in X / -(2x_1 + 2x_2) \leq -4.5\}$$

This resulting program (see Section 3.1) is:

$$\begin{cases} \max \lambda \\ \lambda \leq \frac{5x_1 + 4x_2 - 5}{5} \\ \lambda \leq 1 - (x_1 + x_2 - 2) \\ \lambda \leq 1 - \frac{(-2x_1 - 2x_2 + 9)}{4.5} \\ 5x_1 + 4x_2 \geq 5 \\ x_1 + x_2 \leq 3 \\ x_1 + x_2 \geq 2.25 \\ x_1 \geq 0, x_2 \geq 0, \lambda \geq 0 \end{cases}$$

This program can be merely written:

$$\begin{aligned} \max \lambda \\ \lambda - x_1 - 0.8x_2 &\leq -1 \\ \lambda + x_1 + x_2 &\leq 3 \\ \lambda - 0.44x_1 - 0.44x_2 &\leq -1 \\ -5x_1 - 4x_2 &\leq -5 \\ x_1 + x_2 &\leq 3 \\ -x_1 - x_2 &\leq -2.25 \\ \lambda \geq 0, x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Which yields using LINDO software the solution:

$$x_1 = 2.777778$$

$$x_2 = 0.000000$$

$$\lambda = 0.222222$$

As can be seen, our approach has reconciled inconsistent constraints to deliver a satisfying solution

$x^* = (2.777778, 0.000000)$ and the maximum degree of overall satisfaction of constraints is 22%.

This degree may be improved by a less stringent choice of

d_i ($i = 0, 1, 2$) by the Decision Maker.

Case of Unboundedness of the Feasible Set

Consider now the following linear program

$$(P_u) \begin{cases} \max (x_1 - 4x_2) \\ -2x_1 + x_2 \leq -1 \\ -x_1 - 2x_2 \leq -2 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

In this program we can set x_2 to zero and x_1 arbitrary large. As long as x_1 is greater than 2, the solution will be feasible and it gets large, as the objective function does too. Hence, the solution of this linear program is unbounded. We are going to apply the method described in section 3.2 in a way to find a satisfying solution in line with Decision Maker's preferences. The relaxation program associated with

(P_u) is

$$(P_u)'' \begin{cases} \max (x_1 - 4x_2) \\ -2x_1 + x_2 \leq -1 \\ -x_1 - 2x_2 \leq -2 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

or

$$(P_u)''' \begin{cases} \text{Find } x \in X = \{x = (x_1, x_2) / x_1 \geq 0, x_2 \geq 0\} \\ \text{such that} \\ (x_1 - 4x_2) \gtrsim b_0 \quad (0) \\ -2x_1 + x_2 \lesssim -1 \quad (1) \\ -x_1 - 2x_2 \lesssim -2 \quad (2) \end{cases}$$

Assume b_0 is fixed to 20 by the Decision Maker. Suppose also that subjectively fixed constants for admissible violation

corresponding to flexible constraints (0), (1), and (2) are

$d_0 = 10$, $d_1 = 2$ and $d_2 = 4$ respectively. Then the

membership functions associated to (0), (1) and (2) are respectively:

$$\mu_0(x) = \begin{cases} 1 & \text{if } (x_1 - 4x_2) \geq 20 \\ \frac{(x_1 - 4x_2) - 10}{20 - 10} & \text{if } 10 \leq x_1 - 4x_2 < 20 \\ 0 & \text{if } (x_1 - 4x_2) < 10 \end{cases}$$

$$\mu_1(x) = \begin{cases} 1 & \text{if } -2x_1 + x_2 \leq -1 \\ 1 - \frac{(-2x_1 + x_2) + 1}{(-1 + 2) + 1} & \text{if } -1 < -2x_1 + x_2 \leq 1 \\ 0 & \text{if } (-2x_1 + x_2) > 1 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & \text{if } -x_1 - 2x_2 \leq -2 \\ 1 - \frac{(-x_1 - 2x_2) + 2}{(-2 + 4) + 2} & \text{if } -2 \leq -x_1 - 2x_2 \leq 2 \\ 0 & \text{if } (-x_1 - 2x_2) > 2 \end{cases}$$

The counterpart of $(P3)^{iv}$ for this example is:

$$\begin{aligned} \max \quad & \lambda \\ \lambda - 0.1x_1 + 0.4x_2 \quad & \leq -1 \\ \lambda - x_1 + 0.5x_2 \quad & \leq 0.5 \\ \lambda - 0.25x_1 - 0.5x_2 \quad & \leq 0.5 \\ x_1 - 4x_2 \quad & \geq 10 \\ -2x_1 + x_2 \quad & \leq -1 \\ -x_1 - 2x_2 \quad & \leq -2 \\ \lambda \quad & \leq 1 \\ \lambda \geq 0, x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Using LINDO software, we obtain the satisfying solution $x^* = (20, 0)$ with the maximum degree of overall satisfaction of constraints of 100%, i.e. $\lambda^* = 1$.

Concluding Remarks

Methods for solving linear programming problems should not focus only on regularities. They should also be able to provide guidance on those features that contradict regularities and create complexity, like inconsistency of constraints and unboundedness of the feasible set. These singular features pose challenging questions to the Decision Maker, mostly when he is in need of implementing an action here and now. Dealing with inconsistency and unboundedness in a linear programming setting may be regarded as a leap in the dark. Nevertheless there are different ways to leap in the dark. In this paper, we have adopted a constructivist position to obtain a satisfying solution in a turbulent environment where either inconsistency or unboundedness is in the state of affairs. The grim preamble to the construction of such a satisfying solution is the capture of the Decision Makers views and opinions. Fuzzy sets theory has been used both as a language for representing Decision Maker's subjective opinions and as a tool for combining them. Extensions of this work can explore various directions. It is worth mentioning that using the min operator to translate the meaning of the intersection may be a too pessimistic approach. In decision problems where such an ultra pessimist attitude cannot be accepted,

compensatory operators can be used. For instance μ_D can be defined as follows.

$$\mu_D(x) = \gamma \min_i \mu_i(x) + (1-\gamma) \min \left(1, \sum_i \mu_i(x) \right)$$

Where $\gamma \in (0,1)$ is a coefficient of compensation [14]. Moreover, other membership functions that suit better the situation at hand may be considered in lieu and place of piecewise linear functions described here. The most used membership functions include hyperbolic, hyperbolic inverse, logistics and cubic functions (H. Rommelfangger, 1996). A related problem, important but much more difficult is to ponder distances to inconsistency and unboundedness (M. J Canovas, 2006) in a way to advise the Decision maker's during the decision process. This problem deserves more thoughts and further consideration. This will become a topic for future research.

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